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# Quasi-Symmetric Balanced Incomplete Block Designs

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## ABSTRACT

A balanced incomplete block design $(v, b, r, k, \lambda)$ is called quasi-symmetric if each block intersects one other block in $x$ varieties and the remaining $b - 2$ blocks in $y$ varieties each. We show that there are only two families of such designs:

(a) designs formed by two copies of $(v, v, k, k, \lambda)$;

(b) designs with parameters $(4y, 8y - 2, 4y - 1, 2y, 2y - 1)$.

A more general problem is suggested.

## 1. INTRODUCTION

A balanced incomplete block design with parameter set $(v, b, r, k, \lambda)$ is an arrangement of $v$ varieties in $b$ blocks such that each block contains $k < v$ distinct varieties, each variety occurs in $r$ blocks, and every pair of varieties occurs in $\lambda$ blocks. It is well known that $bk = rv$ and $\lambda(v - 1) = r(k - 1)$. Fisher [1] proved that $b \geqslant v$, and this inequality has been strengthened by Stanton and Sprott [6] to show that if there are $\alpha + 1$ identical blocks $(\alpha > 0)$, then $b \geqslant (\alpha + 1) v - (\alpha - 1)$.

We name the blocks of the design $B_1, B_2, \ldots, B_b$, and let $b_{ij}$ denote the number of varieties $B_i \cap B_j$. Proceeding as in [4], [5] and [6], for example, we obtain (with $b_j = b_{1j}$):

$$b_2 + b_3 + \cdots + b_b = k(r - 1), \tag{1}$$

$$b_2^2 + b_3^2 + \cdots + b_b^2 = k(k\lambda - k - \lambda + r), \tag{2}$$

$$(b - 1) \Sigma(b_j - \bar{b})^2 = (r - k)(v - k)(r - \lambda). \tag{3}$$

If $r = k$ then $b = v$ and the BIBD is symmetric. It is well known that in this case $b_{ij} = \lambda$ $(i \neq j)$. This result follows from (3), for if $r = k$ then $b_j = \bar{b}$ for all $j$, and (1) implies $b_j = \lambda$. Conversely, if $b_j = \bar{b}$ for all $j$, then (3) implies that $r = k$ and the design is symmetric.

In this paper we shall study *quasi-symmetric designs*—designs with the property that, for some $x$, $y$ with $x \neq y$, each block intersects one other block in $x$ varieties and the other $b - 2$ blocks in $y$ varieties each. (If $x = y$, the design is symmetric.) It follows immediately from the definition that in a quasi-symmetric design $b$ is even. In the following sections we determine the possible parameter sets of quasi-symmetric designs, and discuss their construction.

## 2. Block Intersection Properties

Suppose that a BIBD contains a block $B_1$ which intersects $s$ other blocks in $x$ varieties each, and the $t$ remaining blocks in $y$ varieties each. Then

$$s + t = b - 1, \tag{4}$$

and (1), (2), and (3) may be rewritten:

$$sx + ty = k(r - 1), \tag{5}$$

$$sx^2 + ty^2 = k(k\lambda - k - \lambda + r), \tag{6}$$

$$(b - 1)\, \Sigma(b_j - \bar{b})^2 = st(x - y)^2. \tag{7}$$

If all blocks have identical intersection properties, we call the design *s-quasi-symmetric*.

For quasi-symmetry (1-quasi-symmetry), $s = 1$ and $t = b - 2$. In a certain sense, the quasi-symmetric designs are as close to symmetric as possible; if all block intersection numbers $b_j$ were the same, the design would be symmetric. Instead, for each block there is exactly one intersection number $x$ which is not equal to the common value $y$ of the other $b - 2$ intersection numbers.

One might attempt to obtain the parameter sets of quasi-symmetric designs from the above relationships, but the computation involved is cumbersome. We may simplify the discussion by using the block-intersection determinant. The BIBD may be represented by its $b$ by $v$ incidence matrix $A = (a_{ij})$, where $a_{ij} = 1$ if $B_i$ contains the $j$-th variety, and $a_{ij} = 0$ otherwise (see, for example, Ryser [2]). Each row of $A$ contains $k$ ones, and each column contains $r$ ones. It is clear from the definition of $A$ that

$$AA' = (b_{ij}), \tag{8}$$

where $b_{ii} = k$ and $b_{ij} = x$ or $y$ for $i \neq j$.

There is no loss in generality numbering the blocks so that $B_1$ and $B_2$, $B_3$ and $B_4$, etc., are the blocks with $x$ varieties in common.

Then

$$AA' = \begin{bmatrix} k & x & y & y & \cdots & y & y \\ x & k & y & y & \cdots & y & y \\ y & y & k & x & \cdots & y & y \\ y & y & x & k & \cdots & y & y \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y & y & y & y & \cdots & k & x \\ y & y & y & y & \cdots & x & k \end{bmatrix}. \tag{9}$$

The determinant of $AA'$ is evaluated by adding all rows to row 1 and removing the common factor $k + x + (b - 2)\,y$, then subtracting $y$ times the first row from all other rows. We obtain

$$\det AA' = [k + x + (b - 2)\,y](k - x)^{b/2}(k + x - 2y)^{(b-2)/2}. \tag{10}$$

However, since the design is symmetric $b > v$, and the rank of $A$ cannot exceed $v$. Thus $AA'$ is a $b$ by $b$ matrix with rank at most $v < b$, and therefore $\det AA' = 0$. Since $k \geqslant 1$ and $b \geqslant 2$, $k + x + (b - 2)\,y \neq 0$, and we have the following result:

THEOREM 1. *In a quasi-symmetric design, either* $x = k$ *or*

$$x + k - 2y = 0.$$

The first possibility noted in Theorem 1 is easily discussed. If $x = k$, then $B_1 = B_2$, $B_3 = B_4$, etc., and thus

THEOREM 2. *The quasi-symmetric designs* $(v, 2v, 2k, k, 2\lambda)$ *with* $x = k$ *are obtained by taking two copies of symmetric designs* $(v, v, k, k, \lambda)$.

## 3. THE NON-TRIVIAL QUASI-SYMMETRIC DESIGNS

We now consider the more difficult question of determining the designs which satisfy the second equation of Theorem 1,

$$x + k - 2y = 0. \tag{11}$$

From (5) we have immediately that

$$x + (b - 2)\,y = k(r - 1). \tag{12}$$

Subtracting (11) from (12) and dividing by $b$ we obtain

$$y = kr/b = k^2/v, \tag{13}$$

$$x = 2y - k = k(2k - v)/v. \tag{14}$$

Returning to (6) and substituting for $x$ from (14) gives

$$k(k\lambda - k - \lambda + r) = (b + 2)\, y^2 - 4ky + k^2.$$

Substituting for $k^2$ from (13) we obtain

$$k(r - \lambda) + vy(\lambda - 1) = (b + 2)\, y^2 - 4ky + vy.$$

Next we eliminate $\lambda v$ using $\lambda(v - 1) = r(k - 1)$ to obtain

$$k(r - \lambda) = (b + 2)\, y^2 + (2v + r - \lambda - rk - 4k)\, y.$$

Upon rearranging terms have

$$(r - \lambda)(k - \lambda) = (b + 2)\, y^2 + (2v - rk - 4k)\, y,$$

and substituting for $y$ from (13) gives

$$(r - \lambda)(v - k)\, k/v = [(b + 2)\, k^2/v + 2v - rk - 4k]\, k^2/v.$$

This equation simplifies to

$$v(r - \lambda)(v - k) = 2k(v - k)^2.$$

However, in a BIBD $v > k$ and thus

$$v(r - \lambda) = 2k(v - k). \tag{15}$$

But

$$v - k = (v - 1) - (k - 1) = \lambda^{-1}r(k - 1) - (k - 1) = \lambda^{-1}(r - \lambda)(k - 1),$$

and (15) becomes

$$\lambda v(r - \lambda) = 2k(r - \lambda)(k - 1).$$

Since $r - \lambda > 0$, we have

$$\lambda v = 2k(k - 1).$$

Now $\lambda = r(k - 1)/(v - 1)$ and $k - 1 > 0$ imply that

$$vr = 2k(v - 1). \tag{16}$$

Finally from $bk = rv$ we have

$$b = 2(v - 1). \tag{17}$$

It is now easy to determine all parameter sets satisfying (11). By (16), $v$ must divide $2k$. However $v > k$, and thus $v = 2k$. Now (17) gives $b = 4k - 2$, and $bk = rv$ implies that $r = 2k - 1$; $\lambda = k - 1$. But from (13), $y = k^2/v = k/2$ and thus $k = 2y$. We have proved

THEOREM 3. *The non-trivial quasi-symmetric designs have parameters*

$$v = 4y, \qquad b = 8y - 2, \qquad r = 4y - 1, \qquad k = 2y, \qquad \lambda = 2y - 1.$$

Note that the designs of Theorem 3 have $x = 0$ and thus contain $q$ pairs of complementary blocks. In fact they form a subclass $H_2(2y - 1)$ of the family of designs discussed in [6]. Sprott [3, series 4] has used Bose's second module theorem to construct such designs when $v = 4y - 1$ is a prime power. We have

THEOREM 4. *The quasi-symmetric designs of theorem* 3 *exist for infinitely many values of* $y$,

For example, to form a quasi-symmetric design (8, 14, 7, 4, 3), we start with the Fano geometry and adjoin a new symbol $\infty$ to each block. The remaining 7 blocks are obtained by taking complements of these blocks with respect to the set of all 8 varieties. We have

$$\begin{array}{ccccccc} 124\infty, & 235\infty, & 346\infty, & 457\infty, & 561\infty, & 672\infty, & 713\infty, \\ 3567, & 4671, & 5712, & 6123, & 7234, & 1345, & 2456. \end{array}$$

This particular quasi-symmetric design is unique (see [5]).

## 4. SUMMARY

We have defined quasi-symmetric designs to be those for which each block intersects one other block in $x$ varieties, and the $b - 2$ remaining blocks in $y$ varieties each. We have shown that there are only two families of such designs:

(a) designs formed by two copies of $(v, v, k, k, \lambda)$,

(b) designs with parameters $(4y, 8y - 2, 4y - 1, 2y, 2y - 1)$.

In Section 2 we defined $s$-quasi-symmetry. Interesting results may be obtained for $s$ small or for $s$ near its maximal value $(b - 1)/2$, and these will be reported later. The present paper has given a discussion of 1-quasi-symmetry.

## References


1. R. A. Fisher, An Examination of the Different Possible Solutions of a Problem in Incomplete Blocks, *Ann. Eugenics* **10** (1940), 52–75.
2. H. J. Ryser, *Combinatorial Mathematics*, Wiley, New York, 1963.
3. D. A. Sprott, Some Series of Balanced Incomplete Block Designs, *Sankhyā* **17** (1956), 185–192.
4. R. G. Stanton, Some Types of Statistical Designs, Canad. Math. Congress, 1963.
5. R. G. Stanton and R. C. Mullin, Uniqueness Theorems in Balanced Incomplete Block Designs, *J. Combinatorial Theory*, to appear.
6. R. G. Stanton and D. A. Sprott, Block Intersections in Incomplete Block Designs, *Canad. Math. Bull.* **7** (1964), 539–48.